Symplectic twistor operator on $\mathbb{R}^{2n}$ and the Segal-Shale-Weil representation

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Abstract

The aim of our article is the study of solution space of the symplectic twistor operator T_s in symplectic spin geometry on standard symplectic space $(\mathbb{R}^{2n}, \omega)$, which is the symplectic analogue of the twistor operator in (pseudo)Riemannian spin geometry. In particular, we observe a substantial difference between the case $n = 1$ of real dimension 2 and the case of $\mathbb{R}^{2n}$, $n > 1$. For $n > 1$, the solution space of T_s is isomorphic to the Segal-Shale-Weil representation.

Key words: Symplectic twistor operator, Symplectic Dirac operator, Metaplectic Howe duality.

MSC classification: 53C27, 53D05, 81R25.

1 Introduction and Motivation

In the case when the second Stiefel-Whitney class of a Riemannian manifold is trivial, there is a double cover of the frame bundle and consequently there is an associated vector bundle for the spinor representation of the spin structure group. There are two basic first order invariant differential operators acting on spinor valued fields, namely the Dirac operator and the twistor operator. Their spectral properties are reflected in the geometrical properties of the underlying manifold. In Riemannian geometry, the twistor equation appeared as an integrability condition for the canonical almost complex structure on the twistor space, and it plays a prominent role in conformal differential geometry due to its larger symmetry group. In physics, its solution space defines infinitesimal isometries in Riemannian supergeometry. For an exposition with panorama of examples, cf. [6], [1] and references therein.

The symplectic version of Dirac operator D_s was introduced in [10], and its differential geometric properties were studied in [4], [8], [9]. The metaplectic Howe duality for D_s was introduced in [2], allowing to characterize the space of solutions for the symplectic Dirac operator on the (standard) symplectic space $(\mathbb{R}^{2n}, \omega)$.

The aim of the present article is to study the symplectic twistor operator T_s in context of the metaplectic Howe duality, and consequently to determine its solution space on the standard symplectic space $(\mathbb{R}^{2n}, \omega)$. These operators were considered from different perspective in [9], [11], [12]. From an analytic point of view, T_s is represented by overdetermined

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system of partial differential equations and acts on the space of polynomials valued in the vector space of the Segal-Shale-Weil representation. From the point of view of representation theory, T_s is $mp(2n, \mathbb{R})$-invariant and our problem is dissolved by proper understanding of the interaction of T_s with generators D_s, X_s of the Howe dual Lie algebra $sl(2)$.

As we shall see, concerning T_s there is a substantial difference between the situation for $n = 1$ and $n > 1$. Namely, there is an infinite number of irreducible $mp(2n, \mathbb{R})$-modules with different infinitesimal character for $n = 1$, while for $n > 1$ the kernel contains just the Segal-Shale-Weil representation, a result of independent interest. This is the reason why we decided to treat the case $n = 1$ in a separate paper [5] using different, more combinatorial approach, which will be useful in complete understanding of the full infinite dimensional symmetry group of our operator.

The structure of our article is as follows. In the first section we review the subject of symplectic spin geometry and metaplectic Howe duality. In the second section, we start with the definition of the symplectic twistor operator T_s and compute the space of polynomial solutions of T_s on $\mathbb{R}^{2n}$. These results follow from the careful study of algebraic and differential consequences of T_s. In the last third section we indicate the collection of unsolved problems directly related to the topic of the present article.

Throughout the article, we use the notation $\mathbb{N}_0$ for the set of natural numbers including zero, and $\mathbb{N}$ for the set of natural numbers without zero.

1.1 Metaplectic Lie algebra $mp(2n, \mathbb{R})$, symplectic Clifford algebra and a class of simple lowest weight modules for $mp(2n, \mathbb{R})$

In the present section we recall several algebraic and representation theoretical results used in the next section for the analysis of the solution space of the symplectic twistor operator T_s, see e.g., [2], [4], [7], [8], [9].

Let us consider $2n$-dimensional symplectic vector space $(\mathbb{R}^{2n}, \omega = \sum_{i=1}^n e_i^\star \wedge e_{n+i}^\star, n \in \mathbb{N}$, and a symplectic basis $\{e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}\}$ with respect to the non-degenerate two form $\omega \in \wedge^2(\mathbb{R}^{2n})^\star$. Let $E_{k,j}$ be the $2n \times 2n$ matrix with 1 on the intersection of the k-th row and the j-th column and zero otherwise. The set of matrices

$$X_{kj} = E_{k,j} - E_{n+k,n+j},\, Y_{kj} = E_{k,n+j} + E_{j,n+k},\, Z_{kj} = E_{n+k,j} + E_{n+j,k},$$

for $j, k = 1, \ldots, n$ is a basis of $sp(2n, \mathbb{R})$, and can be realized by first order differential operators

$$X_{kj} = x_j\partial_k - x_{n+j}\partial_{n+j},\, Y_{kj} = x_{n+j}\partial_k + x_{n+k}\partial_j,\, Z_{kj} = x_j\partial_{n+k} + x_k\partial_{n+j}.$$

The metaplectic Lie algebra $mp(2n, \mathbb{R})$ is the Lie algebra of the twofold group cover $\pi : Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$ of the symplectic Lie group $Sp(2n, \mathbb{R})$. It can be realized by homogeneity two elements in the symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$, where the homomorphism

$$\pi_\star : mp(2n, \mathbb{R}) \to sp(2n, \mathbb{R})$$

is given by

$$\pi_\star(e_k \cdot e_j) = -Y_{kj},$$
$$\pi_\star(e_{n+k} \cdot e_{n+j}) = Z_{kj},$$
$$\pi_\star(e_k \cdot e_{n+j} + e_{n+j} \cdot e_k) = 2X_{kj}, \tag{1}$$

for $j, k = 1, \ldots, n$.

Definition 1.1 *The symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$ is an associative unital algebra over $\mathbb{C}$, given by quotient of the tensor algebra $T(e_1, \ldots, e_{2n})$ by a two-sided ideal $< I > \subset T(e_1, \ldots, e_{2n})$ generated by*

$$v_j \cdot v_k - v_k \cdot v_j = -i\omega(v_j, v_k)$$

for all $v_j, v_k \in \mathbb{R}^{2n}$, where $i \in \mathbb{C}$ is the complex unit.

The symplectic Clifford algebra $Cl_s(\mathbb{R}^{2n}, \omega)$ is isomorphic to the Weyl algebra W_{2n} of complex valued algebraic differential operators on $\mathbb{R}^n$, and the symplectic Lie algebra $sp(2n, \mathbb{R})$ can be realized as a subalgebra of W_{2n}. In particular, the Weyl algebra is an associative algebra generated by $\{q_1, \ldots, q_n, \partial_{q_1}, \ldots, \partial_{q_n}\}$, the multiplication operator by q_j and differentiation ∂_{q_j}, for $j = 1, \ldots, n$, and the symplectic Lie algebra $sp(2n, \mathbb{R})$ has a basis $\{-\frac{i}{2}q_j^2, -\frac{i}{2}\frac{\partial^2}{\partial q_j^2}, q_j\frac{\partial}{\partial q_j} + \frac{1}{2}\}$, for $j = 1, \ldots, n$.

The symplectic spinor representation is an irreducible Segal-Shale-Weil representation of $Cl_s(\mathbb{R}^{2n}, \omega)$ on $L^2(\mathbb{R}^n, e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$, the space of square integrable functions on $(\mathbb{R}^n, e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$ with $dq_{\mathbb{R}^n}$ the Lebesgue measure. Its action, the symplectic Clifford multiplication c_s, preserves the subspace of C^∞(smooth)-vectors given by Schwartz space $S(\mathbb{R}^n)$ of rapidly decreasing complex valued functions on $\mathbb{R}^n$ as a dense subspace. The space $S(\mathbb{R}^n)$ can be regarded as a smooth (Frechet) globalization of the space of $\tilde{K} = \widetilde{U}(n)$-finite vectors in the representation, where $\tilde{K} \subset Mp(2n, \mathbb{R})$ is the maximal compact subgroup given by the double cover of $K = U(n) \subset Sp(2n, \mathbb{R})$. Though we work in the smooth globalization $S(\mathbb{R}^n)$, the representative vectors are usually chosen to belong to the underlying Harish-Chandra module of $\tilde{K} = \widetilde{U}(n)$-finite vectors preserved by c_s.

The function spaces associated to the Segal-Shale-Weil representation are supported on $\mathbb{R}^n \subset \mathbb{R}^{2n}$, a maximal isotropic subspace of $(\mathbb{R}^{2n}, \omega)$. In its restriction to $mp(2n, \mathbb{R})$, it decomposes into two unitary representations realized on the subspace of even resp. odd functions:

$$\varrho : mp(2n, \mathbb{R}) \rightarrow End(S(\mathbb{R})), \tag{2}$$

where the basis vectors act by

$$\varrho(e_j \cdot e_k) = iq_j q_k,$$
$$\varrho(e_{n+j} \cdot e_{n+k}) = -i\partial_{q_j}\partial_{q_k},$$
$$\varrho(e_j \cdot e_{n+j} + e_{n+k} \cdot e_j) = q_j\partial_{q_{n+k}} + \partial_{q_{n+k}}q_j. \tag{3}$$

for all $j, k = 1, \ldots, n$. In this representation $Cl_s(\mathbb{R}^{2n}, \omega)$ acts on $L^2(\mathbb{R}^n, e^{-\frac{1}{2}\sum_{j=1}^n q_j^2} dq_{\mathbb{R}^n})$ by continuous unbounded operators with domain $S(\mathbb{R}^n)$. The space of

$\tilde{K} = \widetilde{U}(n)$-finite vectors consists of even resp. odd homogeneity $mp(2n, \mathbb{R})$-submodule

$$\{Pol_{even}(q_1, \dots, q_n)e^{-\frac{1}{2}\sum_{j=1}^n q_j^2}\}, \quad \{Pol_{odd}(q_1, \dots, q_n)e^{-\frac{1}{2}\sum_{j=1}^n q_j^2}\}.$$

It is also an irreducible representation of $mp(2n, \mathbb{R}) \ltimes h(n)$, the semidirect product of $mp(2n, \mathbb{R})$ and $(2n + 1)$-dimensional Heisenberg Lie algebra spanned by $\{e_1, \dots, e_{2n}, Id\}$. In the article we denote the Segal-Shale-Weil representation by $\mathcal{S}$, and $\mathcal{S} \simeq \mathcal{S}_+ \oplus \mathcal{S}_-$ as $mp(2n, \mathbb{R})$-module.

Let us denote by $Pol(\mathbb{R}^{2n})$ the vector space of complex valued polynomials on $\mathbb{R}^{2n}$, and by $Pol_l(\mathbb{R}^{2n})$ the subspace of homogeneity l polynomials. The complex vector space $Pol_l(\mathbb{R}^{2n})$ is as an irreducible $mp(2n, \mathbb{R})$-module isomorphic to $S^l(\mathbb{C}^{2n})$, the l-th symmetric power of the complexification of the fundamental vector representation $\mathbb{R}^{2n}$, $l \in \mathbb{N}_0$.

1.2 Segal-Shale-Weil representation and the metaplectic Howe duality

Let us recall a representation-theoretical result of [3], formulated in the opposite convention of highest weight metaplectic modules. Let $\omega_1, \dots, \omega_n$ be the fundamental weights of the Lie algebra $sp(2n, \mathbb{R})$, and let $L(\lambda)$ denotes the simple module over universal enveloping algebra $\mathcal{U}(mp(2n, \mathbb{R}))$ of $mp(2n, \mathbb{R})$ generated by highest weight vector of the weight λ.

Algebraically, the decomposition of the space of polynomial functions on $\mathbb{R}^{2n}$ valued in the Segal-Shale-Weil representation corresponds to the tensor product of $L(-\frac{1}{2}\omega_n)$ resp. $L(\omega_{n-1} - \frac{3}{2}\omega_n)$ with symmetric powers $S^k(\mathbb{C}^{2n})$ of the fundamental vector representation $\mathbb{C}^{2n}$ of $sp(2n, \mathbb{R})$, $k \in \mathbb{N}_0$. The following result is well known.

Corollary 1.2 *([3]) We have for $L(-\frac{1}{2}\omega_n)$*

1. In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side):

$$L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \simeq L(-\frac{1}{2}\omega_n) \oplus L(\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n)$$

$$\oplus L(2\omega_1 - \frac{1}{2}\omega_n) \oplus L(3\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \dots$$

$$\oplus L((2l-1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(2l\omega_1 - \frac{1}{2}\omega_n),$$

2. In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side):

$$L(-\frac{1}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \simeq L(\omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(\omega_1 - \frac{1}{2}\omega_n)$$

$$\oplus L(2\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(3\omega_1 - \frac{1}{2}\omega_n) \oplus \dots$$

$$\oplus L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus L((2l+1)\omega_1 - \frac{1}{2}\omega_n),$$

We have for $L(\omega_{n-1} - \frac{3}{2}\omega_n)$

1. *In the even case $k = 2l$ ($2l + 1$ terms on the right-hand side):*

$$L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \simeq L(\omega_{n-1} - \frac{3}{2}\omega_n) \oplus L(\omega_1 - \frac{1}{2}\omega_n)$$

$$\oplus L(2\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \cdots \oplus L((2l-1)\omega_1 - \frac{1}{2}\omega_n)$$

$$\oplus L(2l\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n),$$

2. *In the odd case $k = 2l + 1$ ($2l + 2$ terms on the right-hand side):*

$$L(\omega_{n-1} - \frac{3}{2}\omega_n) \otimes S^k(\mathbb{C}^{2n}) \simeq L(-\frac{1}{2}\omega_n) \oplus L(\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n) \oplus \ldots$$

$$\oplus L(2l\omega_1 - \frac{1}{2}\omega_n) \oplus L((2l+1)\omega_1 + \omega_{n-1} - \frac{3}{2}\omega_n).$$

A more geometrical reformulation of this statement is realized in the algebraic (polynomial) Weyl algebra and termed metaplectic Howe duality, [2]. The metaplectic analogue of the classical theorem on the separation of variables allows to decompose the space $Pol(\mathbb{R}^{2n}) \otimes \mathcal{S}$ of complex polynomials valued in the Segal-Shale-Weil representation under the action of $mp(2n, \mathbb{R})$ into a direct sum of simple lowest weight $mp(2n, \mathbb{R})$-modules

$$Pol(\mathbb{R}^{2n}) \otimes \mathcal{S} \simeq \bigoplus_{l=0}^{\infty} \bigoplus_{j=0}^{\infty} X_s^j M_l, \tag{4}$$

where we use the notation $M_l := M_l^+ \oplus M_l^-$. This decomposition takes the form of an infinite triangle

$$
\begin{array}{ccccccccccc}
P_0 \otimes \mathcal{S} & & P_1 \otimes \mathcal{S} & & P_2 \otimes \mathcal{S} & & P_3 \otimes \mathcal{S} & & P_4 \otimes \mathcal{S} & & P_5 \otimes \mathcal{S} \\
\| & & \| & & \| & & \| & & \| & & \| \\
M_0 & \longrightarrow & X_s M_0 & \longrightarrow & X_s^2 M_0 & \longrightarrow & X_s^3 M_0 & \longrightarrow & X_s^4 M_0 & \longrightarrow & X_s^5 M_0 \\
& & \oplus & & \oplus & & \oplus & & \oplus & & \oplus \\
& & M_1 & \longrightarrow & X_s M_1 & \longrightarrow & X_s^2 M_1 & \longrightarrow & X_s^3 M_1 & \longrightarrow & X_s^4 M_1 \\
& & & & \oplus & & \oplus & & \oplus & & \oplus \\
& & & & M_2 & \longrightarrow & X_s M_2 & \longrightarrow & X_s^2 M_2 & \longrightarrow & X_s^3 M_2 \\
& & & & & & \oplus & & \oplus & & \oplus \\
& & & & & & M_3 & \longrightarrow & X_s M_3 & \longrightarrow & X_s^2 M_3 \\
& & & & & & & & \oplus & & \oplus \\
& & & & & & & & M_4 & \longrightarrow & X_s M_4 \\
& & & & & & & & & & \oplus \\
& & & & & & & & & & M_5
\end{array}
\tag{5}
$$

Let us now explain the notation used on the previous picture. First of all, we used the shorthand notation $P_l = Pol_l(\mathbb{R}^{2n})$, $l \in \mathbb{N}_0$, and all spaces and arrows on the picture have the following meaning. Let $i \in \mathbb{C}$

be the complex unit. The three operators

$$X_s = \sum_{j=1}^{n} (x_{n+j}\partial_{q_j} + ix_j q_j),$$

$$D_s = \sum_{j=1}^{n} (iq_j \partial_{x_{n+j}} - \partial_{x_j}\partial_{q_j}),$$

$$E_s = \sum_{j=1}^{2n} x_j \partial_{x_j}, \tag{6}$$

where D_s acts on the previous picture horizontally as X_s but in the opposite direction, fulfill the $sl(2)$-commutation relations:

$$[E_s + n, D_s] = -D_s,$$
$$[E_s + n, X_s] = X_s, \tag{7}$$
$$[X_s, D_s] = i(E + n).$$

For the purposes of our article, we do not need the proper normalization of the generators D_s, X_s, E_s making the isomorphism with standard commutation relations in $sl(2)$ explicit.

The elements of $Pol(\mathbb{R}^{2n}) \otimes \mathcal{S}$ are called polynomial symplectic spinors. Let $s \equiv s(x_1, \ldots, x_{2n}, q_1, \ldots, q_n) \in Pol(\mathbb{R}^{2n}) \otimes \mathcal{S}$, $h \in \mathrm{Mp}(2n, \mathbb{R})$ and $\pi(h) = g \in \mathrm{Sp}(2n, \mathbb{R})$ for the double cover map $\pi : \mathrm{Mp}(2n, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$. We define the action of $Mp(2n, \mathbb{R})$ on $Pol(\mathbb{R}^{2n}) \otimes \mathcal{S}$ by

$$\tilde{\varrho}(h)s(x_1, \ldots, x_{2n}, q_1, \ldots, q_n) = \varrho(h)s(\pi(g^{-1})(x_1, \ldots, x_{2n})^T, q_1, \ldots, q_n), \quad (8)$$

with ϱ acting on the Segal-Shale-Weil representation via (2). Passing to the infinitesimal action, we get the operators representing the basis elements of $mp(2n, \mathbb{R})$. For example, we have for $j = 1, \ldots, n$

$$\tilde{\varrho}(X_{jj})s = \frac{d}{dt}\Big|_{t=0} \tilde{\varrho}(\exp(tX_{jj}))s(x_1, \ldots, x_{2n}, q_1, \ldots, q_n)$$
$$= \frac{d}{dt}\Big|_{t=0} e^{\frac{t}{2}} s(x_1, \ldots, x_j e^{-t}, \ldots, x_{n+j}e^t \ldots, x_{2n}, q_1, \ldots, q_j e^t, \ldots, q_n)$$
$$= \Big(\frac{1}{2} - x_j \frac{\partial}{\partial x_j} + x_{n+j}\frac{\partial}{\partial x_{n+j}} + q_j \frac{\partial}{\partial q_j}\Big)s(x_1, \ldots, x_{2n}, q_1, \ldots, q_n).$$

These operators satisfy the commutation rules of the Lie algebra $mp(2n, \mathbb{R})$, and preserve the homogeneity in $x_1, \ldots, x_{2n}$. The operators X_s and D_s commute with operators $\tilde{\varrho}(X_{jk}), \tilde{\varrho}(Y_{jk})$ and $\tilde{\varrho}(Z_{jk})$, $j, k = 1, \ldots, n$, hence are $mp(2n, \mathbb{R})$ intertwining differential operators.

The action of $mp(2n, \mathbb{R}) \times sl(2)$ generates the multiplicity free decomposition of $Pol(\mathbb{R}^{2n}) \otimes \mathcal{S}$ and the pair of Lie algebras in the product is called metaplectic Howe dual pair. The operators X_s, D_s acting in the previous picture horizontally, isomorphically identify the two neighboring $mp(2n, \mathbb{R})$-modules. The modules M_l, $l \in \mathbb{N}$ on the most left diagonal of our picture are termed symplectic monogenics, and are characterized as l-homogeneous solutions of the symplectic Dirac operator D_s. Thus the decomposition is given as a vector space by tensor product of the symplectic monogenics multiplied by polynomial algebra of invariants $\mathbb{C}[X_s]$.

2 The symplectic twistor operator T_s and its solution space on $\mathbb{R}^{2n}$

We start with an abstract definition of the symplectic twistor operator T_s. Let (M, ω) be a $2n$-dimensional symplectic manifold, $\pi : P \to M$ a principal fiber $Sp(2n, \mathbb{R})$-bundle of symplectic frames on M. A metaplectic structure on (M, ω) is a principal fiber $Mp(2n, \mathbb{R})$-bundle $\tilde{P} \to M$ together with bundle morphism $\tilde{P} \to P$, equivariant with respect to the double covering map $Mp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})$. The manifold (M, ω) with a metaplectic structure is usually called symplectic spin manifold. The symplectic manifold M admits a metaplectic structure if and only if the second Stiefel-Whitney class $w_2(M)$ is trivial, and the equivalence classes of metaplectic structures are classified by $H_1(M, \mathbb{Z}_2)$. On $(\mathbb{R}^{2n}, \omega)$ there is a unique metaplectic structure.

Definition 2.1 *Let (M, ∇, ω) be a symplectic spin manifold of dimension $2n$, ∇^s the associated symplectic spin covariant derivative and $\omega \in C^\infty(M, \wedge^2 T^\star M)$ a non-degenerate 2-form such that $\nabla \omega = 0$. We denote by $\{e_1, \ldots, e_{2n}\}$ a local symplectic frame. The symplectic twistor operator T_s on M is the first order differential operator T_s acting on smooth symplectic spinors $\mathcal{S}$:*

$$\nabla^s : C^\infty(M, \mathcal{S}) \longrightarrow T^\star M \otimes C^\infty(M, \mathcal{S}),$$
$$T_s := P_{Ker(c)} \circ \omega^{-1} \circ \nabla^s : C^\infty(M, \mathcal{S}) \longrightarrow C^\infty(M, \mathcal{T}), \qquad (9)$$

where $\mathcal{T}$ is the space of symplectic twistors, $T^\star M \otimes \mathcal{S} \simeq \mathcal{S} \oplus \mathcal{T}$, given by algebraic projection

$$P_{Ker(c_s)} : T^\star M \otimes C^\infty(M, \mathcal{S}) \longrightarrow C^\infty(M, \mathcal{T})$$

on the kernel of the symplectic Clifford multiplication c_s. In the local symplectic coframe $\{\epsilon^1\}_{j=1}^{2n}$ dual to the symplectic frame $\{e_j\}_{j=1}^{2n}$ with respect to ω, we have the local formula for T_s:

$$T_s = \sum_{k=1}^{2n} \epsilon^k \otimes \nabla^s_{e_k} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \nabla^s_{e_k}, \qquad (10)$$

where $\cdot$ is the shorthand notation for the symplectic Clifford multiplication and $i \in \mathbb{C}$ is the imaginary unit. We use the convention $\omega^{kj} = 1$ for $j = k + n$ and $k = 1, \ldots, n$, $\omega^{kj} = -1$ for $k = n+1, \ldots, 2n$ and $j = k - n$, and $\omega^{kj} = 0$ otherwise.

The symplectic Dirac operator D_s is defined as the image of the symplectic Clifford multiplication c_s, and a symplectic spinor in the kernel of D_s is called symplectic monogenic.

Lemma 2.2 *The symplectic twistor operator T_s is $mp(2n, \mathbb{R})$-invariant.*

Proof: The property of invariance is a direct consequence of the equivariance of symplectic covariant derivative and invariance of algebraic projection $P_{Ker(c_s)}$, and amounts to show that

$$T_s(\tilde{\varrho}(g)s) = \tilde{\varrho}(g)(T_s s) \qquad (11)$$

for any $g \in mp(2n, \mathbb{R})$ and $s \in C^\infty(M, \mathcal{S})$. Using the local formula (10) for T_s in a local chart $(x_1, \ldots, x_{2n})$, both sides of (11) are equal

$$\sum_{k=1}^{2n} \epsilon^k \otimes \varrho(g) \frac{\partial}{\partial x_k} \big[s\big(\pi(g)^{-1}x\big) \big]$$

$$+ \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \Big[\varrho(g) \frac{\partial}{\partial x_k} \big[s\big(\pi(g)^{-1}x\big) \big] \Big]$$

and the proof follows.

$\square$

In the case $M = (\mathbb{R}^{2n}, \omega)$, the symplectic Dirac and the symplectic twistor operators are given by

$$D_s = \sum_{j,k=1}^{2n} \omega^{kj} e_k \cdot \frac{\partial}{\partial x_j}, \tag{12}$$

$$T_s = \sum_{l=1}^{2n} \epsilon^l \otimes \frac{\partial}{\partial x_l} + \frac{i}{n} \sum_{j,k,l=1}^{2n} \epsilon^l \otimes \omega^{kj} e_l \cdot e_j \cdot \frac{\partial}{\partial x_k} = \sum_{l=1}^{2n} \epsilon^l \otimes \Big(\frac{\partial}{\partial x_l} - \frac{i}{n} e_l \cdot D_s \Big), \tag{13}$$

and we restrict their action to the space of polynomial symplectic spinors.

Lemma 2.3 *Let $s \in Pol(\mathbb{R}^{2n}, \mathcal{S}^n)$ be a symplectic spinor in the solution space of the symplectic twistor operator T_s. Then s is in the kernel of the square of the symplectic Dirac operator D_s^2.*

Proof: Let s be a polynomial symplectic spinor in $\mathrm{Ker}(T_s)$,

$$T_s s = \sum_{l=1}^{2n} \epsilon^l \otimes \Big(\frac{\partial}{\partial x^l} - \frac{i}{n} e_l \cdot D_s \Big) s = 0, \tag{14}$$

i.e.

$$\Big(\frac{\partial}{\partial x^l} - \frac{i}{n} e_l \cdot D_s \Big) s = 0, \quad l = 1, \ldots, 2n. \tag{15}$$

We apply to the last equation partial differentiation $\frac{\partial}{\partial x^m}$, multiply it by the skew symmetric form ω^{ml} and sum over $m = 1, \ldots, 2n$:

$$\sum_{l,m=1}^{2n} \Big(\omega^{ml} \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^l} - \frac{i}{n} \omega^{ml} e_l \cdot \frac{\partial}{\partial x^m} D_s \Big) s = 0. \tag{16}$$

The first part is zero because of the skew-symmetry resp. symmetry in m, l, and the second part is (a non-zero multiple of) the square of the symplectic Dirac operator D_s^2. Hence

$$\sum_{l,m=1}^{2n} \frac{i}{n} \omega^{ml} e_l \cdot \frac{\partial}{\partial x^m} D_s s = \frac{i}{n} D_s^2 s = 0 \tag{17}$$

and the proof is complete.

$\square$

Let us consider $mp(2n, \mathbb{R})$-submodules in the split composition series

$$\{0\} \subset Ker(D_s) \subset Ker(D_s^2) \tag{18}$$

with

$$Ker(D_s^2) \simeq Ker(D_s) \oplus Ker(D_s^2)/Ker(D_s), \tag{19}$$

and discuss which of them are in the solution space of the symplectic twistor operator T_s. We have

$$Ker(T_s) = (Ker(T_s) \cap Ker(D_s)) \oplus (Ker(T_s) \cap (Ker(D_s^2)/Ker(D_s))). \tag{20}$$

Lemma 2.4 *Let $n \in \mathbb{N}$ and $s \in Pol(\mathbb{R}^{2n}, \mathcal{S})$ be a symplectic spinor fulfilling*

$$s \in Ker(T_s) \cap Ker(D_s). \tag{21}$$

Then s is a constant (i.e., independent of $x_1, \ldots, x_{2n}$) symplectic monogenic spinor. This is encapsulated on the following picture:

- *$Pol(\mathbb{R}^{2n}, \mathcal{S}_-)$:*

$$\boxed{M_0^-} \longrightarrow X_s M_0^- \longrightarrow X_s^2 M_0^- \longrightarrow X_s^3 M_0^- \longrightarrow \ldots \tag{22}$$
$$\oplus \qquad\qquad \oplus \qquad\qquad \oplus$$
$$M_1^- \longrightarrow X_s M_1^- \longrightarrow X_s^2 M_1^- \longrightarrow \ldots$$
$$\oplus \qquad\qquad \oplus$$
$$M_2^- \longrightarrow X_s M_2^- \longrightarrow \ldots$$
$$\oplus$$
$$M_3^- \longrightarrow \ldots$$

- *$Pol(\mathbb{R}^{2n}, \mathcal{S}_+)$:*

$$\boxed{M_0^+} \longrightarrow X_s M_0^+ \longrightarrow X_s^2 M_0^+ \longrightarrow X_s^3 M_0^+ \longrightarrow \ldots \tag{23}$$
$$\oplus \qquad\qquad \oplus \qquad\qquad \oplus$$
$$M_1^+ \longrightarrow X_s M_1^+ \longrightarrow X_s^2 M_1^+ \longrightarrow \ldots$$
$$\oplus \qquad\qquad \oplus$$
$$M_2^+ \longrightarrow X_s M_2^+ \longrightarrow \ldots$$
$$\oplus$$
$$M_3^+ \longrightarrow \ldots$$

Proof: Let $s \in Pol(\mathbb{R}^{2n}, \mathcal{S})$ be a solution of the symplectic twistor operator, see (15),

$$\Big(\frac{\partial}{\partial x^l} - \frac{i}{n} e_l \cdot D_s\Big) s = 0, \quad l = 1, \ldots, 2n,$$

and at the same time $s \in Ker(D_s)$. This implies

$$\frac{\partial}{\partial x^l} s = 0, \quad l = 1, \ldots, 2n,$$

so s is a constant symplectic spinor. The proof is complete.

$\square$

Lemma 2.5 *Let $s \in Pol(\mathbb{R}^{2n}, \mathcal{S})$ be a symplectic monogenic spinor of homogeneity $h \in \mathbb{N}_0$, i.e. $D_s(s) = 0$. Then the symplectic spinor $X_s(s)$ has the following property:*

1. *If $n = 1$, then $X_s(s)$ is in the kernel of T_s for any homogeneity $h \in \mathbb{N}_0$. This is encapsulated on the following picture:*

   - *$Pol(\mathbb{R}^2, \mathcal{S}_-)$:*

$$
\begin{array}{ccccccccc}
M_0^- & \longrightarrow & \boxed{X_s M_0^-} & \longrightarrow & X_s^2 M_0^- & \longrightarrow & X_s^3 M_0^- & \longrightarrow & \ldots \quad (24)\\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^- & \longrightarrow & \boxed{X_s M_1^-} & \longrightarrow & X_s^2 M_1^- & \longrightarrow & \ldots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^- & \longrightarrow & \boxed{X_s M_2^-} & \longrightarrow & \ldots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^- & \longrightarrow & \ldots
\end{array}
$$

   - *$Pol(\mathbb{R}^2, \mathcal{S}_+)$:*

$$
\begin{array}{ccccccccc}
M_0^+ & \longrightarrow & \boxed{X_s M_0^+} & \longrightarrow & X_s^2 M_0^+ & \longrightarrow & X_s^3 M_0^+ & \longrightarrow & \ldots \quad (25)\\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^+ & \longrightarrow & \boxed{X_s M_1^+} & \longrightarrow & X_s^2 M_1^+ & \longrightarrow & \ldots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^+ & \longrightarrow & \boxed{X_s M_2^+} & \longrightarrow & \ldots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^+ & \longrightarrow & \ldots
\end{array}
$$

2. *If $n > 1$, then $X_s(s)$ is in the kernel of T_s if and only if the homogeneity of s is equal to $h = 0$. This is encapsulated on the following picture:*

   - *$Pol(\mathbb{R}^{2n}, \mathcal{S}_-)$:*

$$
\begin{array}{ccccccccc}
M_0^- & \longrightarrow & \boxed{X_s M_0^-} & \longrightarrow & X_s^2 M_0^- & \longrightarrow & X_s^3 M_0^- & \longrightarrow & \ldots \quad (26)\\
 & & \oplus & & \oplus & & \oplus & & \\
 & & M_1^- & \longrightarrow & X_s M_1^- & \longrightarrow & X_s^2 M_1^- & \longrightarrow & \ldots \\
 & & & & \oplus & & \oplus & & \\
 & & & & M_2^- & \longrightarrow & X_s M_2^- & \longrightarrow & \ldots \\
 & & & & & & \oplus & & \\
 & & & & & & M_3^- & \longrightarrow & \ldots
\end{array}
$$

- $Pol(\mathbb{R}^{2n}, \mathcal{S}_+)$:

$$M_0^+ \longrightarrow \boxed{X_s M_0^+} \longrightarrow X_s^2 M_0^+ \longrightarrow X_s^3 M_0^+ \longrightarrow \ldots \quad (27)$$
$$\oplus \qquad\quad \oplus \qquad\quad \oplus$$
$$M_1^+ \longrightarrow X_s M_1^+ \longrightarrow X_s^2 M_1^+ \longrightarrow \ldots$$
$$\oplus \qquad\quad \oplus$$
$$M_2^+ \longrightarrow X_s M_2^+ \longrightarrow \ldots$$
$$\oplus$$
$$M_3^+ \longrightarrow \ldots$$

Proof: Let s be a non-zero symplectic spinor in the kernel of D_s. The question is when the system of differential equations on s,

$$(\partial_k - \frac{i}{n} e_k \cdot D_s) X_s s = 0, \qquad (28)$$

is fulfilled for all $k = 1, ..., 2n$. In other words we ask when $X_s(s)$ is in the kernel of the symplectic twistor operator. Let us multiply the k-th equation in this system by x_k and sum over all k,

$$(E_s - \frac{i}{n} X_s D_s) X_s s = 0. \qquad (29)$$

We use the $sl(2)$-commutation relations for X_s a D_s resp. E_s a X_s, (7), and the fact that s is in the kernel of D_s. This results in

$$(E_s X_s - \frac{1}{n} X_s E_s - X_s) s = 0. \qquad (30)$$

Assuming that s is of homogeneity h, $E_s s = hs$, the last equation reduces to

$$(h + 1 - \frac{h}{n} - 1) X_s s = h(1 - \frac{1}{n}) X_s s = 0. \qquad (31)$$

Observe that $(1 - \frac{1}{n}) \neq 0$ for $n > 1$, and X_s is $mp(2n, \mathbb{R})$-intertwining map acting injectively on $Pol(\mathbb{R}^{2n}, \mathcal{S})$ as a result of the metaplectic Howe duality (i.e., s is non-zero implies $X_s(s)$ is non-zero.) Because s is assumed to be non-zero, the last display implies that either

1. $h = 0$ and $n \in \mathbb{N}$ is arbitrary, or
2. $n = 1$ and h is arbitrary.

A straightforward check for $n > 1$ and homogeneity $h = 0$ gives

$$(\partial_k - ie_k D_s) X_s s = (e_k + X_s \partial_k - \frac{i}{n} e_k E_s - e_k) s = 0, \qquad (32)$$

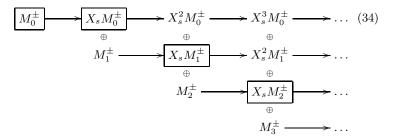and in the case $n = 1$ and arbitrary homogeneity we have

$$(\partial_1 - ie_1 D_s) X_s s = (e_1 + e_1 x_1 \partial_1 + e_2 x_2 \partial_1 - e_1 x_1 \partial_1 - e_1 x_2 \partial_2 - e_1) s$$
$$= (x_2(e_2 \partial_1 - e_1 \partial_2)) s = -x_2 D_s s = 0. \qquad (33)$$

As for the second component $(\partial_{x_2} - ie_2 D_s)$ of the symplectic twistor operator, the computation is analogous to the first one. This completes the proof.

$$\square$$

Let us summarize our results in the final theorem.

Theorem 2.6 *The solution space of the symplectic twistor operator T_s on standard symplectic space $(\mathbb{R}^{2n}, \omega)$ is given by $mp(2n, \mathbb{R})$-modules in the boxes on the following pictures:*

- *In the case $n = 1$, we have for $Pol(\mathbb{R}^2, \mathcal{S}_\pm)$:*

$$
\boxed{M_0^\pm} \longrightarrow \boxed{X_s M_0^\pm} \longrightarrow X_s^2 M_0^\pm \longrightarrow X_s^3 M_0^\pm \longrightarrow \ldots \quad (34)
$$
$$
\oplus \qquad\qquad \oplus \qquad\qquad \oplus
$$
$$
M_1^\pm \longrightarrow \boxed{X_s M_1^\pm} \longrightarrow X_s^2 M_1^\pm \longrightarrow \ldots
$$
$$
\oplus \qquad\qquad \oplus
$$
$$
M_2^\pm \longrightarrow \boxed{X_s M_2^\pm} \longrightarrow \ldots
$$
$$
\oplus
$$
$$
M_3^\pm \longrightarrow \ldots
$$

- *In the case $n > 1$, we have for $Pol(\mathbb{R}^{2n}, \mathcal{S}_\pm)$:*

$$
\boxed{M_0^\pm} \longrightarrow \boxed{X_s M_0^\pm} \longrightarrow X_s^2 M_0^\pm \longrightarrow X_s^3 M_0^\pm \longrightarrow \ldots \quad (35)
$$
$$
\oplus \qquad\qquad \oplus \qquad\qquad \oplus
$$
$$
M_1^\pm \longrightarrow X_s M_1^\pm \longrightarrow X_s^2 M_1^\pm \longrightarrow \ldots
$$
$$
\oplus \qquad\qquad \oplus
$$
$$
M_2^\pm \longrightarrow X_s M_2^\pm \longrightarrow \ldots
$$
$$
\oplus
$$
$$
M_3^\pm \longrightarrow \ldots
$$

An interested reader can easily verify the previous result for $n > 1$ by taking a simple solution s of D_s of homogeneity at least one (it is sufficient to generate such a simple solution from dimension $n = 1$ case) and check that $X_s(s) \notin Ker(T_s)$.

Example 2.7 *In the case $n = 2$ and the homogeneity 2 the symplectic spinor*

$$
s = e^{-\frac{q_1^2 + q_2^2}{2}}(-ix_1 x_2 + x_1 x_4 + x_2 x_3 + ix_3 x_4) \tag{36}
$$

is a solution of D_s. However, $X_s s$ is not a solution of the symplectic twistor operator T_s because, for example, the first and the second components of $T_s X_s(s)$ are nonzero:

$$
(T_s s)^1 = \epsilon^1 \otimes e^{-\frac{q_1^2 + q_2^2}{2}} q_2 (x_2 + ix_4)^2 \neq 0,
$$
$$
(T_s s)^2 = \epsilon^2 \otimes e^{-\frac{q_1^2 + q_2^2}{2}} q_1 (x_1 + ix_3)^2 \neq 0.
$$

It is much harder to verify the result $X_s s \in Ker(T_s)$ for all polynomial symplectic spinors s, $s \in Ker(D_s)$, in the case $n = 1$, and we refer to [5] for a non-trivial combinatorial proof of this assertion.

We would like to emphasize that the kernel of our solution space realizes (for $n > 1$) the Segal-Shale-Weil representation, a prominent $Sp(2n, \mathbb{R})$-module with far-reaching impact on harmonic analysis.

3 Comments and open problems

In the present section we comment on the results achieved in our article.

First of all, notice that in the case of (both even and odd) orthogonal algebras and the spinor representation as an orthogonal analogue of the Segal-Shale-Weil representation, the solution space of the twistor operator for orthogonal Lie algebras on $\mathbb{R}^n$ is given by two copies of the spinor representation, in complete analogy with the symplectic case, see [1] for $n \geq 3$. As for $n = 2$, we were not able to find the required result in the available literature, although we believe it is known to specialists. Here one half of the Dirac operator is the Dolbeault operator and the twistor operator is its complex conjugate, while the opposite halves of the Dirac resp. twistor operators are their complex conjugates. The solution spaces for both halves of the twistor operator on $\mathbb{R}^2$ are complex linear spans of polynomials $\{z^j\}_{j\in\mathbb{N}_0}$ resp. $\{\bar{z}^j\}_{j\in\mathbb{N}_0}$, intersecting non-trivially in the constant polynomials. This is an orthogonal analogue of our results in symplectic category, and indicates an infinite-dimensional symmetry group acting on the solutions spaces of both symplectic Dirac and symplectic twistor operators in the real dimension 2.

Another observation is related to the proof of Lemma 2.3 and its structure on curved symplectic manifolds. Let us consider a $2n$-dimensional metaplectic manifold (M, ∇, ω), with ∇^s the lifted metaplectic covariant derivative. Then a differential consequence of the symplectic twistor equation on M is

$$\sum_{l,m=1}^{2n} \left(\omega^{ml}(\nabla_m^s, \nabla_l^s) - \frac{i}{n}D_s^2 \right)s = 0, \tag{37}$$

where the first term (skewing of the composition of metaplectic covariant derivatives) gives the action of the symplectic curvature of the symplectic connection ∇^s on the space of sections of a metaplectic bundle on M. This equation should be thought of as a symplectic analogue of the equation

$$D^2 s = \frac{1}{4}\frac{n}{n-1}Rs, \quad n \geq 3 \tag{38}$$

in Riemannian spin geometry with s a twistor spinor, D the Dirac operator and R the scalar curvature of the Riemannian structure, cf. [1]. The prolongation of the symplectic twistor equation then constructs a linear connection and covariant derivative on the Segal-Shale-Weil representation, in such a way that the covariantly constant sections correspond to symplectic twistor spinors.

Another ramification of our results is related to the higher order twistor equations acting on symplectic spinors, with leading part

$$s \to \Pi^{hw}\nabla_{(i_1}^s \ldots \nabla_{i_j)}^s s, \tag{39}$$

where Π^{hw} is projection on the highest weight (or, Cartan) component and round bracket denote the symmetric part in symplectic covariant derivatives. For example, their solution spaces can be studied by its interaction with metaplectic Howe duality in an analogous way as we did in the first order case.

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